12. S. P. Levitskii and A. T. Listrov, in: Heat and Mass Transfer VI [in Russian], Vol. 6, Minsk (1980), part 2, pp. 80-88.
13. Z. P. Shul'man and S. P. Levitskii, Dynamics and Heat and Mass Transfer of Bubbles in Relaxing Polymer Clusters. Preprint No. 2 [in Russian], A. V. Lykov Institute of Heat and Mass Transfer, Academy of Sciences of the Belorussian SSR, Minsk (1984).

VORTEX MOTION IN DILUTE POLYMER SOLUTIONS

A. L. Yarin

UDC 532.527:532.135

Vortex-dynamic equations have been derived and applied to two problems concerning annular vortices.

1. It has been found that an important part is played by the behavior of individual vortex structures in turbulent transport [1, 2] from measurements on turbulent boundary layers and friction reduction by polymers (about 0.001-0.01%). Here I present a quasi-one-dimensional approach to vortex dynamics (vortex filaments of finite thickness) in dilute polymer solutions.

Measurements show that a major stage (vortex stretching) can be retarded by the entropy-dependent elasticity in macromolecular coils. The vortex is formed in a boundary layer or by a generator via velocity pulsations or pulses arising from the generator. The vorticity is generated in a shear layer at the boundary of the flow, with the leading part acquiring a mushroom form and being transferred to the core of the vortex [2]. At the head of the flow, the liquid particles are spread out as occurs in a free disk film arising from the collision of a thin jet with a small target [3]. One therefore expects that the vortex will contain macromolecular coils at its core, which are stretched along the axis. If the vortex is then stretched at a rate exceeding θ^{-1}, the longitudinal elastic stresses in the core will increase (otherwise, they will decrease). These stresses influence the velocity pattern and thus the core evolution. The vortex is surrounded by unstretched liquid, and although the liquid and the core contain macromolecules, the elastic stresses in the latter are negligible, and it may be considered as ideal (if vortex diffusion is negligible).

We now consider a vortex whose core is subject to a longitudinal elastic stress (the liquid is considered as incompressible). We assume as a first approximation that this stress is constant over the cross section of the core. Correspondingly, the stress tensor in the core will be $\sigma' = \sigma'_{\tau\tau}\,\tau\tau$. The sum of this elastic tensor σ' and the viscous-stress tensor is the deviator for the stress tensor in the liquid [4]. The equation for the core vorticity is

$$\frac{D\Omega}{Dt} = (\Omega\cdot\nabla)\,\mathbf{v} + \nu\Delta\Omega + \frac{1}{\rho}\nabla\times(\nabla\cdot\sigma'). \tag{1}$$

From (1), viscous effects are unimportant for times less than $t < m^2/\nu$, which we consider.

The contribution from the elastic stresses in (1) is

$$\nabla\times(\nabla\cdot\sigma') = \mathbf{b}\left(\sigma'_{\tau\tau}\frac{\partial k_*}{\partial\xi} + 2k_*\frac{\partial\sigma'_{\tau\tau}}{\partial\xi}\right) - k_*\varkappa_*\sigma'_{\tau\tau}\mathbf{n}. \tag{2}$$

The right side of (2) is zero for rectilinear and annular vortices and is small in the long-wave approximation for any vortex. We therefore restrict ourselves to that approximation for times $t < m^2/\nu$, where the second and third terms on the right in (1) can be

Institute of Mechanics Problems, Academy of Sciences of the USSR, Moscow. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 53, No. 2, pp. 222-229, August, 1987. Original article submitted April 8, 1986.

0022-0841/87/5302-0897$12.50

neglected, after which (1) reduces to a Helmholtz equation. In that approximation, the vortex is frozen, although the core has longitudinal elastic stresses.

In the core,

$$\frac{D\mathbf{v}}{Dt} = -\frac{1}{\rho}\nabla p + \nu\Delta\mathbf{v} + \frac{1}{\rho}\nabla\cdot\sigma'. \tag{3}$$

For times $t < m^2/\nu$, the viscous effects are unimportant, and (3) gives

$$\frac{D\mathbf{v}}{Dt} = -\frac{1}{\rho}\nabla p + \frac{1}{\rho}\left(k_*\mathbf{n}\sigma'_{\tau\tau} + \tau\frac{\partial\sigma'_{\tau\tau}}{\partial\xi}\right). \tag{4}$$

We denote by C the circular contour of the core section; we introduce the polar angle ϕ and an element of C denoted by $d\mathbf{l} = md\phi(\mathbf{n}\cos\phi - \mathbf{b}\sin\phi)$, where (4) gives

$$\oint_C \frac{D\mathbf{v}}{Dt}\cdot d\mathbf{l} = \frac{1}{\rho}k_*\sigma'_{\tau\tau}m\int_0^{2\pi}\cos\varphi d\varphi = 0. \tag{5}$$

Thomson's kinematic theorem [5] gives with (5) that

$$\oint_C \frac{D\mathbf{v}}{Dt}\cdot d\mathbf{l} = \frac{D\Gamma}{Dt} = 0. \tag{6}$$

We then have that the vortices in dilute polymer solutions retain their individuality and intensity Γ.

To trace the vortex evolution, we introduce the radius vector R and velocity V of a liquid particle at the axis, which we parametrize by means of the arbitrary coordinate s:

$$\frac{D\mathbf{R}}{Dt} = \frac{\partial\mathbf{R}}{\partial t} + W\tau = \mathbf{V} = \mathbf{V}_* + \mathbf{w},$$
$$W = \lambda\frac{Ds}{Dt}, \quad \lambda = \left|\frac{\partial\mathbf{R}}{\partial s}\right|, \quad \frac{\partial}{\partial\xi} = \frac{\partial}{\lambda\partial s}. \tag{7}$$

We consider a single vortex in an unbounded ideal liquid immobile at infinity. We use the Biot–Savart formula or the quasi-one-dimensional momentum balance while neglecting terms $o(\ln[1/(k_*m)])$, $k_*m \ll 1$ (thin vortex) to get [6, 7] that

$$\mathbf{V}_* = \frac{\Gamma k_*\mathbf{b}}{4\pi}\ln\frac{1}{k_*m}. \tag{8}$$

An annular vortex in an unbounded ideal liquid is covered by a more accurate result, namely Maxwell's formula, in which terms o(1) are neglected [8]:

$$\mathbf{V}_* = \frac{\Gamma\mathbf{b}}{4\pi a}\left(\ln\frac{8a}{m} - \frac{1}{4}\right). \tag{9}$$

We determine the velocity component w. We consider the vortex as a distinct line and write the quasi-one-dimensional equations of continuity and momentum balance for it as in the theory of thin free liquid jets [9, 10]. That approach has been used previously for motion in heated vortices [11] and for vortices in an ideal liquid showing longitudinal flow within the cores [7, 12].

Individuality in the vortices means that the mass of the liquid in the cores is conserved, which we write $D(\rho f\lambda ds)/Dt = 0$. Then $D(ds)/Dt = \partial(W\lambda^{-1})/\partial s ds$ gives us the quasi-one-dimensional equation of continuity

$$\frac{\partial f\lambda}{\partial t} + \frac{\partial Wf}{\partial s} = 0, \quad f = \pi m^2. \tag{10}$$

The results of [7] are used with the force balance acting on a core element of length λds, which gives

$$\rho_1 \Gamma \boldsymbol{\tau} \times \mathbf{w} + \frac{1}{\lambda} \frac{\partial}{\partial s} (\sigma'_{\tau\tau} f \boldsymbol{\tau}) + (\rho - \rho_1) f \boldsymbol{\tau} \cdot \mathbf{g} \cdot \boldsymbol{\tau} = 0. \tag{11}$$

Here the first term is related to the Zhukovskii force acting on the vortex and the second to the stresses in the core due to the entropy-dependent elasticity in the macromolecular coils; we have also incorporated the mass force acting on the core for liquid of density ρ differing from the density of the surrounding liquid ρ_1.

Typically, w is only a small correction to V_*, and it can then be shown [7] that the forces $\rho f Dw/Dt$ related to the inertia of the core and the surrounding liquid (adjoint mass) are $O((k_* m)^2)$ by comparison with the forces included in (11). The long-wave approximation corresponds to $k_* m \ll 1$ (thin vortices), so inertial forces are not incorporated into (11).

A rheological relation is required as usual to close the equations. Here we can use the Hinch–De Gennes molecular hydrodynamic model [4, 13], which gives as follows for the uniaxial stretching in the core:

$$\sigma'_{\tau\tau} = c\varkappa \frac{N_0 b}{N_0 b - \sqrt{L_{\tau\tau}}} L_{\tau\tau}, \quad \frac{\partial L_{\tau\tau}}{\partial t} + \frac{W}{\lambda} \frac{\partial L_{\tau\tau}}{\partial s} =$$

$$= 2L_{\tau\tau} \left(\frac{1}{\lambda} \frac{\partial V_\tau}{\partial s} - k_* V_n \right) - \frac{2}{\theta} \frac{r_0}{\sqrt{L_{\tau\tau}}} \frac{N_0 b}{N_0 b - \sqrt{L_{\tau\tau}}} \left(L_{\tau\tau} - \frac{r_0^2}{3} \right), \quad r_0 = b N_0^{1/2}, \quad \varkappa = \frac{3kT}{N_0 b^2}. \tag{12}$$

Then the equations describing the individual vortices for dilute polymer solutions are (7) and (8) or (9) with (10)-(12). If there are several vortices and/or an external flow, the contributions from these to the speed of a particular vortex should be added to the right sides of (8) and (9).

2. The first example is a vortex ring in an unbounded ideal liquid moving vertically under buoyancy forces; we use the Lagrange parameter for the axis. We assume that s is equal to the polar angle reckoned in the plane of the ring. Then $\lambda = a$ and $W = 0$ in (7), and (10) gives $am^2 = \text{const}$, while (11) gives

$$\rho_1 \Gamma (\mathbf{e}_r w_z - \mathbf{k} w_r) - \pi m^2 (\rho - \rho_1) g \mathbf{k} = 0. \tag{13}$$

Here it is assumed that the unit vector in the cylindrical coordinate system k and the corresponding axis Oz are directed vertically upwards. From (13) we get that

$$w_z = 0, \quad \rho_1 \Gamma w_r = -\pi m^2 (\rho - \rho_1) g. \tag{14}$$

From the first equation in (7) with (9) and (14) we get

$$\frac{dZ}{dt} = \frac{\Gamma}{4\pi a} \left(\ln \frac{8a}{m} - \frac{1}{4} \right), \quad \frac{da}{dt} = w_r = -\frac{\pi m^2 (\rho - \rho_1) g}{\rho_1 \Gamma}. \tag{15}$$

Also,

$$am^2 = a_0 m_0^2. \tag{16}$$

System (15) and (16) enables one to calculate the motion of a vortex ring. Let $\Gamma < 0$ and $\rho > \rho_1$, which corresponds to the vortex arising when a drop of heavy liquid falls into a light one. Then (15) and (16) imply the paradoxical conclusion predicted in [14]: the vortex moves in the direction of gravity while expanding ($da/dt > 0$) and being retarded (!) approximately as $1/a$.

One mainly uses another approach in calculations on ring vortex motion in the presence of buoyancy forces. The basis is the Helmholz equation in integral form (the equation for the vortex momentum) [8, 14, 15]:

$$\frac{\rho}{2} \frac{D}{Dt} \int \mathbf{r} \times \boldsymbol{\Omega} dv = \int [-(\rho - \rho_1) g \mathbf{k}] dv, \quad \rho \approx \rho_1. \tag{17}$$

Here the integration is taken over the entire infinite volume of liquid.

One calculates the integrals in (17) for the case of an isolated thin annular vortex showing quasisolid motion at the core, which gives the second equation in (15). The first equation in (15) naturally still applies.

This example has been considered to show that there is satisfactory accuracy from (11). There is agreement between the vortex expansion speeds calculated from (17) and (11), which confirms the latter, since (17) differs from (11) in being exact for an ideal liquid. This leads one to hope that (11) is reasonably accurate when one cannot use the approach via the integral Helmholtz equation.

3. These equations are applied to a vortex ring in a polymer solution encountering a parallel wall (a situation similar to the model experiments of [2], in which however the vortices approached the wall at an angle). The condition at the wall is satisfied by introducing a mirror vortex that induces a velocity V_{*1} at the axis of the incoming vortex in accordance with the Biot–Savart formula, where the components along the normal and binormal to the axis are

$$(V_{*1})_n = -\frac{\Gamma Z k_1^3}{4\pi a^2}\left[\left(1-\frac{2}{k_1^2}\right)\frac{E(k_1)}{1-k_1^2}+\frac{2}{k_1^2}K(k_1)\right],$$

$$(V_{*1})_b = \frac{\Gamma k_1}{4\pi a}\left\{\left(1-\frac{2}{k_1^2}\right)K(k_1)+\frac{2}{k_1^2}E(k_1)+\frac{k_1^2Z^2}{a^2}\left[\left(1-\frac{2}{k_1^2}\right)\frac{E(k_1)}{1-k_1^2}+\frac{2}{k_1^2}K(k_1)\right]\right\}, \tag{18}$$

$$k_1 = \frac{1}{\sqrt{1+Z^2/a^2}}.$$

These expressions contain the elliptic integrals

$$E(k_1)=\int_0^{\pi/2}\sqrt{1-k_1^2\sin^2\psi}\,d\psi, \quad K(k_1)=\int_0^{\pi/2}\frac{d\psi}{\sqrt{1-k_1^2\sin^2\psi}}, \tag{19}$$

and $Z > 0$ denotes the distance from the vortex to the wall ($z = 0$).

We assume the following parameters: initial radius of core axis and cross sectional radius of core $a_0 = 10^{-2}$ m and $m_0 = 0.2 \times 10^{-2}$ m, $r_0 = 10^{-7}$ m, $c = 10^{20}$ $1/m^3$, $\rho = \rho_1 = 10^3$ kg/m^3, $\kappa = 10^{-6}$ kg/sec^2, $N_0 = 2.36 \times 10^4$, $|\Gamma| = 10^{-3}$ m^2/sec, $\theta = 10^{-1}$ sec, and $\nu = 10^{-5}$ m^2/sec. These values correspond approximately to the conditions of [2], where the vortices were produced by a generator and the polymer concentrations were 0.02-0.1%. In general, there are links between the macromolecular coils at such concentrations, which increase the viscosity and relaxation time. This circumstance is incorporated here from the choice of θ and ν, which exceed the characteristic values for truly dilute aqueous solutions by an order of magnitude.

The value of m_0^2/ν in this case is 0.4 sec, so we consider times $t < 0.4$ sec to neglect the vortex diffusion. In the experiments of [2], annular vortices were used with radii of curvature for the core axes $a \approx 10^{-2}$ m, and the vortices could travel a distance of about $\sim(1\text{-}2)\cdot 10^{-2}$ m, which was approximately the path length before collision with the wall.

We parametrize the core axis with the same Lagrange parameter as in section 2; then (7), (10), (11), and (12) reduce to the following on the basis of (9):

$$\frac{dZ}{dt}=\frac{\Gamma}{4\pi a}\left(\ln\frac{8a}{m}-\frac{1}{4}\right)+(V_{*1})_b+w_b \quad (\Gamma<0),$$

$$\frac{da}{dt}=-(V_{*1})_n-w_n, \quad am^2=a_0m_0^2, \tag{20}$$

$$w_n=0, \quad w_b=\frac{c\varkappa N_0 b L_{\tau\tau}\pi m^2}{\rho\Gamma a(N_0 b-\sqrt{L_{\tau\tau}})},$$

$$\frac{dL_{\tau\tau}}{dt}=\frac{2L_{\tau\tau}}{a}\frac{da}{dt}-\frac{2}{\theta}\frac{r_0}{\sqrt{L_{\tau\tau}}}\frac{N_0 b}{N_0 b-\sqrt{L_{\tau\tau}}}\left(L_{\tau\tau}-\frac{r_0^2}{3}\right).$$

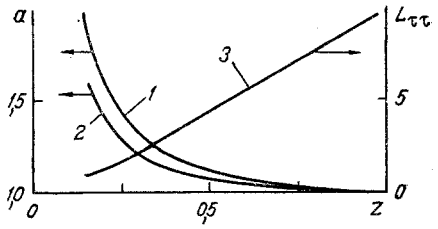

Fig. 1. Changes in the radius of the vortex core axis and in the axial component of the orientation-deformation tensor as a vortex approaches the wall: 1) vortex radius change in an ideal liquid; 2) the same in a polymer solution ($L_{\tau\tau 0} = 10\ r_0^2$); 3) change in $L_{\tau\tau}$ for a vortex in polymer solution (corresponds to curve 2); with Z and a referred to a_0 and $L_{\tau\tau} - r_0^2$, calculation carried through to contact between the wall and the vortex core.

System (18)-(20) was integrated numerically with the initial conditions

$$t = 0, \quad Z = a_0, \quad a = a_0, \quad L_{\tau\tau} = L_{\tau\tau 0}. \tag{21}$$

Figure 1 shows the calculations, which showed that the reduction in the radius of the vortex axis in the polymer solution by 20% near the wall (by comparison with an ideal liquid), which was approximately as in [2], was attained for $L_{\tau\tau 0} = 10\ r_0^2$, which corresponds to initial coil stretching along the circular axis of the vortex to an extent about 5.5 times the equilibrium value, which is here due to vortex formation. The calculations show that the vortex stretching on approaching the wall is slight in the sense that the stretching rate is insufficient to increase the deformation of the macromolecular coils in the axial direction. This is evident from curve 3, where relaxation predominates over stretching and $L_{\tau\tau}$ falls. Only near the wall, where the stretching rate increases, does the rate of decrease in $L_{\tau\tau}$ fall. Nevertheless, the residual unrelaxed elastic stresses are sufficient to retard the stretching by an amount close to that observed in [2].

NOTATION

θ, relaxation time; n, b, τ, unit principal normal, unit binormal, and unit tangent vectors of vortex core axis; $\sigma'_{\tau\tau}$, elastic stress in vortex core cross section; σ', elastic stress tensor; Ω, vorticity vector; D/Dt, derivative with respect to time t; v, velocity vector; ν, ρ, kinematic viscosity and density of the liquid; m, radius of vortex core cross section; k, κ_*, curvature and torsion of vortex core axis; ξ, arc length of vortex core axis; p, pressure; ϕ, polar angle in vortex core cross section; dl, vector element of vortex core cross section contour; Γ, vortex intensity; R, V, radius vector and velocity of particle on vortex core axis; s, arbitrary parameter of vortex core axis; V_*, velocity component of vortex core axis particle induced by other vortices (if any) and by the bulk potential flow; w, velocity component of vortex core axis particle originating from macromolecular elastic entropy forces and weight, for a denser liquid in an immiscible one; a, radius of curvature of vortex ring core; f, area of vortex core cross section; g, acceleration due to gravity; ρ_1, density of surrounding liquid; b, Kuhn segment length; N_0, number of Kuhn segments in macromolecule; r, equilibrium radius of gyration; c, number density of macromolecules; κ, entropy elasticity of macromolecular coil; k, Boltzmann's constant; T, temperature; $L_{\tau\tau}$, axial component of orientation-deformation tensor; e_r, k, unit radial and axial vectors in cylindrical coordinate system; Z, vortex ring position; r, radius vector of particles; dv, volume element; V_{*1}, velocity induced in vortex axis by an imaginary vortex; z, coordinate aong the vortex axis. Subscripts indicate the projections on the corresponding axes. Subscript 0 for values at t = 0.

LITERATURE CITED

1. S. J. Kline, W. C. Reynolds, F. A. Schraub, and P. W. Runstadler, J. Fluid Mech., 30, No. 4, 741-776 (1967).
2. R. E. Falko and D. K. Wigert, Reducing Viscous Friction [Russian translation], Moscow (1984), pp. 285-301.
3. G. Taylor, Proc. R. Soc. London, A253, No. 1274, 296-312 (1959).

4. E. J. Hinch, Phys. Fluids, 20, No. 10, Part 2, 22-30 (1977).
5. N. E. Kochin, I. A. Kibel', and N. V. Roze, Theoretical Hydromechanics [in Russian], Vol. 1, Moscow (1963).
6. R. J. Arms and F. R. Hama, Phys. Fluids, 8, No. 4, 553-559 (1965).
7. D. W. Moore and P. G. Saffman, Phil. Trans. R. Soc. London, 272, No. 1226, 403-429 (1972).
8. H. Lamb, Hydrodynamics, Dover (1932).
9. V. M. Entov and A. L. Yarin, Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gaza, No. 5, 11-18 (1980).
10. V. M. Entov and A. L. Yarin, J. Fluid Mech., 140, 91-111 (1984).
11. A. T. Onufriev, Zh. Prikl. Mekh. Tekh. Fiz., No. 2, 3-15 (1967).
12. S. E. Widnall and D. B. Bliss, J. Fluid Mech., 50, No. 2, 335-353 (1971).
13. P. G. De Gennes, J. Chem. Phys., 60, No. 12, 5030-5042 (1974).
14. M. A. Lavrent'ev and B. V. Shabat, Mathematical Models and Problems in Hydrodynamics [in Russian], Moscow (1973).
15. J. Batchelor, Introduction to Liquid Dynamics [Russian translation], Moscow (1973).

EFFECTS OF VARIABLE PHYSICAL PROPERTIES ON HEAT TRANSFER IN FREE CONVECTION AROUND A HORIZONTAL CYLINDER

G. G. Shklover and S. E. Gusev

UDC 536.25

Theoretical and experimental studies have been made on the effects of variable viscosity and compressibility on the average heat-transfer coefficient.

General formulas have been recommended [2, 3] in [1] for calculating heat transfer by free convection from a horizontal cylinder; the purpose was to obtain a single formula applicable over wide ranges in the Rayleigh and Prandtl numbers. The variability in the physical properties was incorporated by taking the mean boundary-layer temperature $T_v = 0.5(T_0 + T_c)$ as the definitive quantity. Here we determine the ranges in the physical parameters in which these formulas apply. An approximate method is proposed for incorporating the property variation more precisely.

There are comparatively few papers dealing with the effects of variability in physical properties on heat transfer from horizontal cyliners.

Akagi [4] derived an approximate self-modeling solution and showed that the variability can be incorporated by solving a self-modeling equation system analogous to that for a vertical plate. The solution was derived numerically for an exponential temperature dependence of the dynamic viscosity. The average Nusselt number was derived [4] as

$$\mathrm{Nu} = 0.515\,(\mathrm{Gr}_0\mathrm{Pr}_0)^{0.25}\,(\mu_0/\mu_c)^{0.21}, \tag{1}$$

where $\mathrm{Nu} = \alpha D/\lambda$; $\mathrm{Gr}_0 = g\beta_0 D^3(T_c - T_0)/\nu_0^2$, which applies for $\mathrm{Pr}_0 = 100$ to 10000; $\mu_0/\mu_c = 1$ to 140. The correction factor $(\mu_0/\mu_c)^{0.21}$ can be discarded if one calculates the physical properties from the defining temperature $T_e = T_0 + 0.75(T_c - T_0)$. Formula (1) is close to the experimental one derived in [5].

In [6], measurements for the range $10^2 < \nu_0/\nu_c < 10^4$ gave

$$\mathrm{Nu} = 0.70\,(\mathrm{Gr}_0\mathrm{Pr}_0)^{0.25}\,(\nu_0/\nu_c)^{0.14}. \tag{2}$$

Studies have been made [7, 8] on the effects of simultaneous temperature dependence for the bulk expansion coefficient and kinematic viscosity.

Tsiolkovskii Kaluga State Pedagogic Institute. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 53, No. 2, pp. 229-236, August, 1987. Original article submitted June 23, 1986.

 0022-0841/87/5302-0902$12.50